Journal of Statistical Physics, Vol. 47, Nos. 5/6, 1987

The Interface of the Ising Model and the Brownian Sheet

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Received October 30, 1986; revision received January 7, 1987

We study the limit theorem related to the interface of the three-dimensional Ising model. Dobrushin proved that the interface does not fluctuate and becomes rigid for sufficiently large β. We define the random field $X^L(t, s)$, $0 \leqslant t$, $s \leqslant 1$, on the interface, and prove that $X^L(t, s)$ converges to the Brownian sheet as $L \to \infty$ for sufficiently large β, where L denotes the size of the system. This result does not mean that the interface itself converges to the Brownian sheet.

KEY WORDS: Interface; Ising model; standard wall; Brownian sheet.

1. INTRODUCTION

Consider the three-dimensional Ising model in a box, $V = V_{L,M}$,

$$V_{L,M} = \{t \in \mathbb{Z}^3; 0 \leqslant t_1, t_2 \leqslant L, -M \leqslant t_3 \leqslant M+1\}$$

A configuration space in V is given by

$$\Omega_V = \{+1, -1\}^V$$

Let us consider the boundary condition $\omega_\pm$ given by

$$\omega_\pm(t) = \begin{cases} +1 & \text{if } t_3 > 0, \\ -1 & \text{if } t_3 \leqslant 0, \end{cases} \qquad t \in \mathbb{Z}^3 \setminus V$$

We associate to each configuration $\xi \in \Omega_V$ the interaction energy with the boundary condition $\omega_\pm$,

$$H_V(\xi \mid \omega_\pm) = -J \sum_{i,j \in V; |i-j|=1} \xi(i)\,\xi(j) - J \sum_{\substack{i \in V, j \in V^c \\ |i-j|=1}} \xi(i)\,\omega_\pm(j), \qquad J > 0$$

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The Gibbs state on Ω_V for $H_V(\xi \mid \omega_\pm)$ is defined by

$$P_{V,\pm}(\xi) = \frac{1}{Z_{V,\pm}} \exp[-\beta H_V(\xi \mid \omega_\pm)] \tag{1.1}$$

where $\beta > 0$ is the reciprocal temperature.

Due to the boundary condition $\omega_\pm$, there exists a surface λ by which the box V is decomposed into an upper region V_λ^u surrounded by $(+)$ spins and a lower region V_λ^l surrounded by $(-)$ spins. Such a surface λ is called an interface. As in the two-dimensional case, it is convenient to describe $\xi \in \Omega_V$ by a family of contours $(\Gamma_1,..., \Gamma_k)$ in $V_\lambda^u \cup V_\lambda^l$ and an interface λ. Setting $2J = 1$, one rewrites the probability distribution (1.1) in the form

$$P_{V,\pm}(\xi) = \frac{1}{Z_V} \exp\left(-\beta\, |\lambda| - \beta \sum_{i=1}^{k} |\Gamma_i|\right) \tag{1.2}$$

if ξ is described by $(\lambda, \Gamma_1,..., \Gamma_k)$.

The probability distribution of λ is derived from (1.2) as follows:

$$P_{L,M}(\lambda) = \frac{1}{Z_V} \exp(-\beta\, |\lambda|)\, Z_{V_\lambda^u,+} \cdot Z_{V_\lambda^l,-} \tag{1.3}$$

where

$$Z_{V_\lambda^u,+} \quad \text{and} \quad Z_{V_\lambda^l,-}$$

are partition functions in V_λ^u and V_λ^l with $(+)$ and $(-)$ boundary conditions, respectively.

When $\beta = \infty$ the interface is perfectly flat. We call such a perfect flat surface a "standard plane" and denote it by S. The interface λ will be deformed for finite value of β, and this deformation is decomposed into a family of elementary shapes $\mathbf{w} = (w_1,..., w_n)$ called standard walls (see Fig. 1). This notion of standard wall was first introduced by Dobrushin[1] for the study of the existence of non-translation-invariant Gibbs states (see also Ref. 2). Dobrushin proved that for sufficiently large β the interface λ does not fluctuate and becomes "rigid" in the following sense: the probability that the interface passes through a point that is not on S tends to zero as $\beta \to \infty$. This implies the existence of non-translation-invariant Gibbs states for sufficiently large β.

Our main interest is to investigate the fluctuation on this "nearly flat" interface for sufficiently large β by considering some random field.

Before describing our results we shall describe the probability distribution (1.3) in a more convenient form. We use the method of polymer

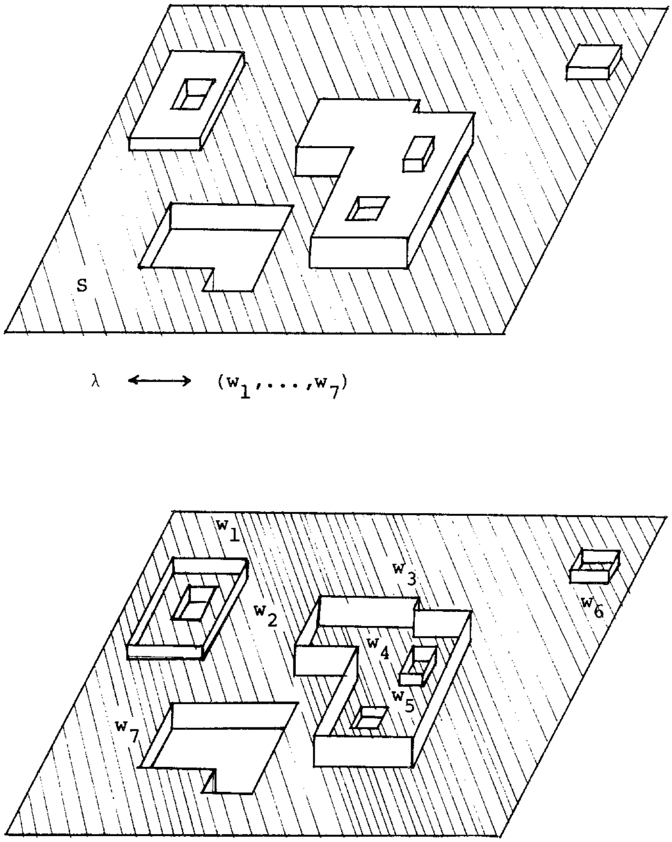

Fig. 1. The representation of λ by the family of standard walls.

expansion developed by Gallavotti *et al.*[3–5] and Del Grosso.[6] Applying the polymer expansion to partition functions

$$Z_{V^u_\lambda,+} \quad \text{and} \quad Z_{V^l_\lambda,-}$$

for sufficiently large β, we get

$$P_{L,M}(\lambda)=\frac{1}{Z_{L,M}}\exp\left[-\beta\sum_{i=1}^{n}|w_i|-U^{\beta}_{V}(w_1,..., w_n)\right] \tag{1.4}$$

when the interface λ is described by the family $\mathbf{w}=(w_1,..., w_n)$ of standard walls. Here, $|w_i|$ is the excess area of w_i and $U^{\beta}_{V}(w_1,..., w_n)$ is the interaction term of $(w_1,..., w_n)$, which is explicitly given in terms of polymer functionals (see Refs. 8 and 9 for details).

If we write $U_V^\beta(w_1,..., w_n)$ as the sum of potentials $\Phi_V^\beta(\cdot)$,

$$U_V(w_1,..., w_n) = \sum_{\substack{\{i_1,...,i_k\} \\ \subset \{1,...,n\}}} \Phi_V^\beta(w_{i_1},..., w_{i_k})$$

then the potential $\Phi_V^\beta(w_1,..., w_k)$ decays exponentially as the mutual distance of the standard walls tends to ∞,

$$|\Phi_V^\beta(w_1,..., w_k)| \leqslant c(\beta) \min_{1 \leqslant i \leqslant k} |w_i| \exp[-c\beta d(w_1,..., w_k)]$$

where $d(w_1,..., w_k)$ is the shortest length of the path connecting all $w_1,..., w_k$.

Hence, the family of standard walls is considered to be the weakly dependent sequence of the random variables if β is sufficiently large.

Letting $M \to \infty$, we have that $P_{L,M}(\lambda)$ weakly converges to the probability distribution $P_L(\lambda)$ of the interface λ in $V_L = \{t \in \mathbb{Z}^3;\ 0 \leqslant t_1, t_2 \leqslant L\}$.

2. STATEMENT OF RESULT

In the previous section we regarded the interface λ as the configuration $\mathbf{w} = (w_1,..., w_n)$ of standard walls.

Now, we shall define the random field $X^L(t, s)(\mathbf{w})$, $0 \leqslant t, s \leqslant 1$. Let $\mathscr{P}$ be the set of all standard walls on S and $T: \mathscr{P} \to \mathscr{P}$ be the mapping that maps every point of w to its mirror image with respect to the standard plane S (see Fig. 2).

We consider the functional $F(w)$ defined for $w \in \mathscr{P}$ and assume the following three conditions on F:

(0) F is not constantly 0, i.e., $F \not\equiv 0$.

(i) $F(Tw) = -F(w)$.

(ii) $|F(w)| < \exp(c_0 |w|)$ for some $c_0 > 0$.

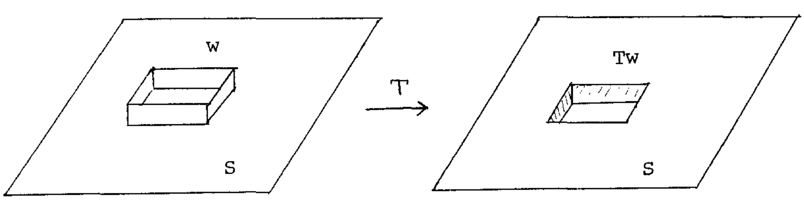

Fig. 2. The transformation T.

For a given family $\mathbf{w}$ of standard walls on S, we define $X^L(t, s)(\mathbf{w})$, $0 \leqslant t, s \leqslant 1$, by

$$X^L(t, s)(\mathbf{w}) = \frac{1}{\sigma L} \sum_{\substack{w \in \mathbf{w}; w \\ \subset [0, tL] \times [0, sL]}} F(w)$$

where $\sigma > 0$ and the sum runs over all $w \in \mathbf{w}$ contained in

$$\{z = (z_1, z_2, \tfrac{1}{2}) \in S; 0 \leqslant z_1 \leqslant tL \text{ and } 0 \leqslant z_2 \leqslant sL\}$$

Theorem. For sufficiently large β there exists a function $\sigma_F(\beta) > 0$, and the finite-dimensional distribution of $X^L(t, s)$ with $\sigma = \sigma_F(\beta)$ converges to the corresponding distribution of the Brownian sheet, i.e.,

$$\begin{aligned} &P_L(X^L(t_i, s_i) \in [T_i, S_i] \quad (i = 1,..., k)) \\ &\quad \to P(W(t_i, s_i) \in [T_i, S_i] \quad (i = 1,..., k)) \qquad \text{as} \quad L \to \infty \end{aligned}$$

where $T_i < S_i$ $(i = 1,..., k)$ and $(W(t, s), P)$ is a Brownian sheet.

For the convenience of the reader we give the definition of Brownian sheet in the following. Let $\mathscr{C}$ be the set of all continuous functions $f(t, s)$ on $[0, 1]^2$ such that $f(t, s) = 0$ if $t = 0$ or $s = 0$. The set A of $[0, 1]^2$ is called a block if it is given in the form $A = (s_1, t_1] \times (s_2, t_2]$. In a similar way to the Brownian motion, we define the increment $W(A)$ for a stochastic process $W = \{W(t, s); (t, s) \in [0, 1]^2\}$ by

$$W(A) = W(t_1, t_2) - W(s_1, t_2) - W(t_1, s_2) + W(s_1, s_2)$$

The Brownian sheet $W = \{W(t, s); (t, s) \in [0, 1]^2\}$ is characterized by the following two conditions:

(i) $P(W \in \mathscr{C}) = 1$.

(ii) If the set of blocks $A_1,..., A_k$ are disjoint, then $W(A_1),..., W(A_k)$ are independent and normally distributed with mean 0 and variances $|A_1|,..., |A_k|$, where $|A_i|$ is the area of the block A_i.

We define functions χ_j^L, $j = 1,..., k$, by

$$\chi_j^L(w) = \begin{cases} 1 & \text{if} \quad w \subset [0, t_j L] \times [0, s_j L] \\ 0 & \text{otherwise} \end{cases}$$

For any $y_1,..., y_k \in \mathbb{R}$, we define the function $f_L = f_L(y_1,..., y_k)$ of $\mathbf{w}$ by

$$f_L(w) = \sum_{i=1}^{k} y_i \sum_{w \in \mathbf{w}} F(w) \chi_i^L(w)$$

Consider the characteristic function $\theta_k^L(\mathbf{y}) = \theta_k^L(y_1,..., y_k)$ of random vectors

$$\left(\frac{1}{\sigma L} \sum F(w)\, \chi_1^L(w),..., \frac{1}{\sigma L} \sum F(w)\, \chi_k^L(w)\right)$$

defined by

$$\theta_k^L(\mathbf{y}) = \langle \exp[if_L(\cdot)/\sigma L] \rangle_{P_L}$$

We prove that $\theta_k^L(\mathbf{y})$ converges to the corresponding characteristic function of the Brownian sheet (see Ref. 9).

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